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Symmetry breaking and semilinear elliptic equations

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1. Introduction

It is a readily observable fact that many physical and mathematical systems possess a degree of symmetry and that a study of this symmetry may give us valuable insight into their behaviour. It is particularly interesting that symmetric systems exist which possess non-symmetric solutions and where this solution branch arises from a symmetry breaking bifurcation on a branch of symmetric solutions. In this paper we shall study an example of such a system. Namely we shall study the symmetry breaking bifurcations (henceforth denoted as SBB's) which occur on the radially symmetric solution branches of the following semilinear elliptic equation.

$$\begin{aligned} &\Delta u + \lambda f(u) = 0 \quad &&\text{for } \boldsymbol{r} \in B, \\ &u = 0 &&\text{for } \boldsymbol{r} \in \partial B. \end{aligned} \tag{1.1}$$

Where B is the unit ball in the space $\mathbb{R}^3$ and Δ is the usual Laplacian operator. It is easily seen that this system is invariant under the action of the rotation group O_3.

In this paper we shall develop a general theory for such systems which permits a straightforward calculation of the location of the SBB's for problem (1.1). This theory will then be applied to numerically study branches of SBB's when the function $f(u)$ takes the special form

$$f(u) = u(1 + |u|^{p-1}) \quad \text{with } p > 1. \tag{1.2}$$

In particular we shall study how the location of the SBB's varies as we change the value of p. Of special interest will be values of p denoted by p^* at which two SBB's coalesce and values denoted by p^∞ at which the problem (1.1)–(1.2) has an infinite number of SBB's.

Definition 1.1. A *radially symmetric solution branch* for problem (1.1) (henceforth denoted by Γ) is a connected set of solutions $(\lambda, u) \in \mathbb{R} \times C^2(B)$ such that the function $u(\boldsymbol{r})$ is invariant under the action of the group O_3 (and thus $u(\boldsymbol{r}) = u(\mathrm{r})$ where $r = |\boldsymbol{r}|$.) A non-symmetric solution branch for problem (1.1) is any connected set of solutions (λ, u) not possessing this invariance property.

Definition 1.2. A *symmetry breaking bifurcation point* (an SBB) for Γ is any point $(\lambda, u) \in \Gamma$ which is a limit point of a non-symmetric solution branch of problem (1.1) or which lies on the intersection of Γ with a non-symmetric solution branch.

(We note that the definition of an SBB given above allows us to consider both pitchfork bifurcations and transcritical bifurcations.)

The SBB's play a significant role in the study of the stability of the symmetric solutions of problem (1.1). A particular instance of this is the change in the solutions if the domain B is perturbed. In general the symmetric solution branch Γ will perturb smoothly but in a neighbourhood of an SBB more complex behaviour is possible. Some of this is described by Chillingworth [6]. We are further motivated to study the location of the SBB's of problem (1.1) if we wish to numerically calculate the non-symmetric solution branches. An effective numerical method for calculating these is to first compute Γ and then to locate the SBB's. As we shall show in this paper, both of these calculations involve solving fairly simple systems or ordinary differential equations. We may then search for non-symmetric solutions in a neighbourhood of the SBB with further local information about these solutions being given by a Liapounov–Schmidt reduction of the system at the SBB.

The branches of solutions bifurcating from the SBB are, in general, symmetric under the action of some subgroup of O_3 and the forms this symmetry may take are described by Ihrig and Golubitsky [11] and by Golubitsky and Schaeffer [9]. It is shown by Vanderbauwhede [15] however, that the problem (1.1) always has a branch of solutions bifurcating from the SBB which is invariant under the action of the group O_2 and this branch is more readily computable.

A necessary condition for the existence of an SBB is that for some $(\lambda, u) \in \Gamma$ there is a solution $\bar{\psi}(\boldsymbol{r})$ of the following linear elliptic boundary value problem.

$$\begin{aligned} &\Delta\bar{\psi} + \lambda f_u(u)\bar{\psi} = 0 \quad &&\text{for } \boldsymbol{r} \in B, \\ &\bar{\psi} = 0 &&\text{for } \boldsymbol{r} \in \partial B. \end{aligned} \tag{1.3}$$

Definition 1.3 (Smoller and Wasserman [14]). If the problem (1.3) has a solution $\bar{\psi}(\boldsymbol{r})$ at some point $(\lambda, u) \in \Gamma$ such that $\bar{\psi}(\boldsymbol{r})$ is not invariant under the action of the group O_3, then there is an infinitesimal symmetry breaking bifurcation (henceforth denoted by ISB) at this point.

The existence of an ISB at the point $(\lambda, u) \in \Gamma$ is certainly not a sufficient condition for the existence of an SBB and a further transversality condition is needed to ensure this. (This is very similar to the transversality condition for a Hopf bifurcation.)

The purpose of this paper is to describe an algorithm for determining the location of the ISB's and for verifying the transversality condition. This algorithm will be supported by both theoretical and numerical examples related to problem (1.2), where we shall show that the behaviour of the ISB's changes greatly as p passes through the value 5 (the critical Sobolev exponent for $\mathbb{R}^3$).

2. The calculation of ISB points

In this section we present an algorithm for calculating the ISB's for the radially symmetric solutions of problem (1.1). Such a solution is a function of r only where $r = |\boldsymbol{r}|$ and hence it satisfies the following ordinary differential equation.

$$u_{rr} + \frac{2}{r}u_r + \lambda f(u) = 0,$$
$$u_r(0) = u(1) = 0. \tag{2.1}$$

We now introduce a rescaling described by Smoller and Wasserman [14] and by Budd and Norbury [5] namely we set

$$\boldsymbol{s} = \lambda^{1/2}\boldsymbol{r}, \quad s \equiv |\boldsymbol{s}| = \lambda^{1/2}r \quad \text{and} \quad v(s) = u(\lambda^{-1/2}s).$$

The function $v(s)$ then satisfies the following ordinary differential equation.

$$v_{ss} + \frac{2}{s}v_s + f(v) = 0, \tag{2.2}$$

and

$$v_s(0) = v(\mu) = 0, \tag{2.3}$$

where $\mu = \lambda^{1/2}$.

We may now readily determine a branch of radially symmetric solutions of problem (2.1) by setting $u(0) = v(0) = N$ and then solving the ordinary differential equation problem (2.2) together with the initial conditions

$$v_s(0) = 0, \qquad v(0) = N. \tag{2.4}$$

(The existence of a bounded solution of this problem is proved in Smoller and Wasserman [14].)

If $\mu_j(N)$ is the jth positive zero of the function $v(s)$ then we say that the corresponding function $u(r)$ lies on the jth symmetric solution branch of problem (1.1).

Similarly, if we set $\psi(\boldsymbol{s}) = \bar{\psi}(\lambda^{-1/2}\boldsymbol{s})$ the partial differential equation problem (1.3) transforms to

$$\begin{aligned} &\Delta\psi + f_u(v)\psi = 0 && \text{for } \boldsymbol{s} \in B_\mu, \\ &\psi = 0 && \text{for } \boldsymbol{s} \in \partial B_\mu. \end{aligned} \tag{2.5}$$

Where $B_\mu = \{\boldsymbol{s} \in \mathbb{R}^3 : s \equiv |\boldsymbol{s}| < \mu \equiv \lambda^{1/2}\}$. (In this paper it will sometimes prove convenient to work with the functions $v(s)$, $\psi(\boldsymbol{s})$ and sometimes with $u(r)$ and $\bar{\psi}(\boldsymbol{r})$. We shall use the above notation for these functions consistently throughout.)

We shall now seek solutions of problem (2.5) in the form

$$\psi(\boldsymbol{s}) = S_l(s)Y_{lm}(\theta, \phi) \tag{2.6}$$

where $\boldsymbol{s}(s, \theta, \phi)$ (in spherical polar notation) and the function $Y_{lm}(\theta, \phi)$ is an lth order Spherical Harmonic. Similarly we shall set

$$\bar{\psi}(\boldsymbol{r}) = T_l(r)Y_{lm}(\theta, \phi) \tag{2.7}$$

where $T_l(r) = S_l(\lambda^{1/2}r)$ and $\boldsymbol{r} = (r, \theta, \phi)$.

From the completeness of the set of Spherical Harmonics and the smoothness of the solutions of problem (2.5) we note that all such solutions must comprise a linear combination of functions of the form (2.6) for integer values of the parameter l. This description of the function $\psi(\boldsymbol{s})$ is intimately related to the standard representation of the group O_3. Indeed the $(2l + 1)$st dimension irreducible representation of this group may be expressed in terms of its action upon the set

$$\{Y_{lm}(\theta, \phi);\ -l \leqslant m \leqslant l\}.$$

The following preliminary results on the behaviour of the function $\psi(s)$ have been established by Smoller and Wasserman [14] and by Budd [1].

Lemma 2.1. (i) *Suppose that* $\mu \equiv \mu_1(N)$ *so that* $u(r) > 0$ *for* $r \in [0,1)$ *and* $f(u) \in C^1(\mathbb{R})$ *with* $f(0) \geqslant 0$. *Then problem* (2.5) *has no solution. Thus there are no SBB's on positive solution branches of problem* (1.1).

(ii) *If problem* (2.5) *has a solution with the function* $\psi(s)$ *defined as in* (2.6) *with* $l = 0$ *then the corresponding bifurcation does not involve a change in the symmetry of the solution and is consequently not an SBB.*

(iii) *If* $f(0) \geqslant 0$ *then problem* (2.5) *has no solutions if* $l = 1$.

It is further shown by Budd [1] and by Budd and Norbury [5] that if $f(u)$ is defined as in (1.2) and if $\mu \equiv \mu_j$ there is a value $L(j) < \infty$ such that if $l > L(j)$ the problem (2.5) has no solution $\psi(s)$ in the form (2.6).

If we substitute the expression (2.6) into the partial differential equation (2.5) we find that the function $S_l(s)$ is the solution of the following ordinary differential equation problem.

$$(S_l)_{ss} + \frac{2}{s}(S_l)_s - l(l+1)/s^2 S_l + f_u(v) S_l = 0, \tag{2.8}$$

$$S_l(\mu) = 0 \quad \text{and} \quad S_l(s) \to 0 \quad \text{as } s \to 0. \tag{2.9}$$

We shall now consider (2.8) as an initial value problem. Namely we define a function $R_l(s)$ to be the solution of the ordinary differential equation (2.8) together with the initial condition

$$R_l(s)/s^l \to 1 \quad \text{as } s \to 0.$$

We now make the following definitions.

Definition 2.1. Let $v(s)$ be the solution of the ordinary differential equation problem with $v(0) = N$ and let $R_l(s)$ be defined as above. We define $\mu_j(N)$ to be the jth positive zero of $v(s)$ and $\alpha_{lk}(N)$ to be the kth positive zero of $R_l(s)$.

We are now in a position to state the main result of this paper.

Lemma 2.2. *Let* $\mu_j(N)$, $\alpha_{lk}(N)$ *be defined as above and let* $f(u) \in C^1(\mathbb{R})$. *Then*

(i) $\mu_j(N)$ *and* $\alpha_{lk}(N)$ *are both differentiable functions of N.*

(ii) *If* $l \leqslant l'$ *and* $k \leqslant k'$ *then* $\alpha_{lk}(N) \leqslant \alpha_{l'k'}(N)$.

(iii) *Iff for some value of the triple* (j, k, l) *there is a value of N such that*

$$\alpha_{lk}(N) = \mu_j(N) \tag{2.10}$$

then there is an ISB for problem (1.1) *with* $\lambda = \mu_j(N)^2$ *and with* $u(0) = N$.

(iv) *If the condition* (2.10) *is satisfied for some values of N and j and there is only one value of l such that this is true then a sufficient condition for an SBB is*

$$\mathrm{d}\big(\alpha_{lk}(N) - \mu_j(N)\big)/\mathrm{d}N \neq 0. \tag{2.11}$$

(The transversality condition (2.11) is similar—and closely related to—the usual transversality condition for a steady state bifurcation described by, for example, Crandall and Rabinowitz [7]. A very similar transversality condition also holds for the rather different problem of a Hopf bifurcation described by Guckenheimer and Holmes [10].)

As an application of Lemma 2.2 we have the following result.

Corollary 2.3. *Suppose that there are values μ_j^0, α_{lk}^0 and μ_j^∞, α_{lk}^∞ such that*

$$\mu_j(N) \to \mu_j^0, \quad \alpha_{lk}(N) \to \alpha_{lk}^0 \quad \text{as } N \to 0$$

and

$$\mu_j(N) \to \mu_j^\infty, \quad \alpha_{lk}(N) \to \alpha_{lk}^\infty \quad \text{as } N \to \infty.$$

Then, if $\mu_j^\infty < \alpha_{lk}^\infty \leqslant \alpha_{lk}^0 < \mu_j^0$ there is an ISB on the jth solution branch of problem (1.1).

(The special case of $\mu_j^\infty = \alpha_{lk}^\infty$ is of some interest and we shall discuss it further in Section 4.)

Proof. From the continuity of the functions $\mu_j(N)$ and $\alpha_{lk}(N)$ we may deduce from the above conditions that there must exist a value of $N = N_1$ such that $0 < N_1 < \infty$ and $\mu_j(N_1) = \alpha_{lk}(N_1)$. Hence we may deduce from (iii) that there is an ISB when $u(0) = N_1$. □

Example 2.1. If $f(u) = u + u^5$ it is shown by Budd and Norbury [5] that if $(j, k, l) = (2, 1, 2)$ there exist values μ_2^0, α_{21}^0 μ_2^∞ and α_{21}^∞ satisfying the above conditions. Further

$$\mu_2^0 = 2\pi, \quad \mu_2^\infty = \tfrac{3}{2}\pi \quad \text{and} \quad \alpha_{21}^0 = \alpha_{21}^\infty = \alpha^*$$

where α^* is the first zero of the Bessel function $J_{5/2}(s)$ and hence $\alpha^* = 5.763459\ldots$. Thus, as $\frac{3}{2}\pi < \alpha^* < 2\pi$ we may deduce the existence of an ISB on this branch. (It may further be shown that this point is an SBB.)

Example 2.2. If $f(u) = u(1 + |u|^{p-1})$, $p > 5$ the above hypotheses may be verified both numerically and by the formal asymptotic methods described by Budd [1]. It may be shown that there are values of $p > 5$ (in particular values close to 5) such that the condition (2.10) is satisfied for some N provided that the triple (j, k, l) assumes the following values.

$j = 1,$	none;	
$j = 2,$	$k = 1,$	$l = 2;$
$j = 3,$	$k = 1,$	$l = 4,$
	$k = 1,$	$l = 5,$
	$k = 2,$	$l = 2;$
$j = 4,$	$k = 1,$	$l = 7,$
	$k = 2,$	$l = 4,$
	$k = 3,$	$l = 2.$

We shall study the nature of the ISB points in the above examples in Sections 3 and 4.

Part of the importance of Lemma 2.2 is the observation that the curves $\mu_j(N)$ and $\alpha_{lk}(N)$ can be easily calculated by using standard integration routines. The conditions (2.10) and (2.11) may then be readily verified and hence we have a simple test for ISB and SBB points. In Section 4 we shall describe a numerical method implementing this idea in more detail. Further, the condition (2.11) is easy to verify numerically and in Section 3 we shall discuss the close connection between this condition and the bifurcation equations of problem (1.1). We shall also present some examples of systems where this condition fails.

We shall now conclude this section by proving Lemma 2.2.

Proof of Lemma 2.2. The result (i) has been proven by Smoller and Wasserman [14].

To prove (ii) we firstly observe (trivially) from the definition of $\alpha_{lk}(N)$ that $\alpha_{lk}(N) < \alpha_{lk'}(N)$ if $k < k'$. Further, the inequality $\alpha_{lk}(N) < \alpha_{l'k}(N)$ if $l < l'$ follows immediately from the Sturmian Comparison Theorem for solutions of ordinary differential equations.

To prove (iii) we see that if $\mu_j(N) = \alpha_{lk}(N)$ the function $R_l(s)$ is a linear multiple of the functions $S_l(s)$ defined by the formulae (2.8) and (2.9) and consequently the partial differential equation (2.5) has a solution of the form (2.6). Conversely, if such a solution exists then $S(\mu_j(N)) = 0$ and consequently

$$\mu_j(N) = \alpha_{lk}(N) \quad \text{for some value of } k.$$

To conclude the proof of Lemma 2.2 it remains for us to establish the condition (2.11) in (iv). To do this we appeal to the following very general theorem due to Vanderbauwhede.

Theorem 2.4 (Vanderbauwhede [15]). *Let the nonlinear operator problem*

$$M(w;\ \lambda) = 0 \tag{2.12}$$

be equivariant under the action of a representation G of the group O_3. Suppose further that $M(0;\ \lambda) \equiv 0$. Let the point $(\lambda,\ w) = (\lambda_0,\ 0)$ be an ISB point and hence the kernel K of the linear operator $M_w(0;\ \lambda_0)$ is not the empty set.

Then, an SBB occurs at the above point if the following two conditions are satisfied.

(i) *The restriction of G to the set K is irreducible and*

(ii) $M_{\lambda w}(0,\ \lambda_0)\zeta \notin$ range $M_w(0;\ \lambda_0)$ *for all functions* $\zeta \in K$.

Further, if an SBB occurs there is a branch of axisymmetric solutions bifurcating from the point $(\lambda,\ w) = (\lambda_0,\ 0)$.

To use this theorem we must reformulate the problem (1.1) in the standard form described in (2.12). To do this we suppose that problem (1.1) has a radially symmetric solution branch $\Gamma \equiv (\lambda,\ u(r))$ where $u(r)$ is a solution of the ordinary differential equation problem (2.1). We now define a map $M(w;\ \lambda) : C^0(B) \to C^0(B)$ as follows.

$$M(w,\ \lambda) \equiv w + u(r) + \lambda\Delta^{-1}f(w + u(r)) = 0 \tag{2.13}$$

where Δ^{-1} is the Green's function for the Laplacian operator on

$$M_w(0;\ \lambda_0)\bar{\psi} \equiv \bar{\psi} + \lambda_0\Delta^{-1}f_u(u(r))\bar{\psi}, \tag{2.14}$$

so that the operator M_w is a compact perturbation of the identity map. It is evident from the standard theory of linear elliptic operators, given for example by Gilbarg and Trudinger [8] that the function $\bar{\psi}(r)$ is a solution of the eigenvalue problem

$$M_w(0;\ \lambda_0) = 0 \tag{2.15}$$

iff it is also a solution of the elliptic boundary value problem (1.3). Hence the function $\bar{\psi}(\boldsymbol{r})$ may be expressed as a linear combination of terms of the form (2.7) where we note that the value of l given in this expression need not be unique. Thus the kernel K of the operator $M_w(0;\ \lambda_0)$ is spanned by the following set

$$K = \text{span}\left[\bigcup_{l \in L} \{T_l(r) Y_{lm}(\theta, \phi):\ -l \leqslant m \leqslant l\}\right] \tag{2.16}$$

where the set L may have several members. If, however, L has only one element then the representation of the group O_3 on the set K takes its standard $(2l+1)$ dimensional form. Hence this representation is irreducible and the condition (ii) of Theorem 2.4 is thus satisfied. The above restriction of the set L follows iff the differential equation problem (2.8) has a solution only for a unique value of l. Hence, if the identity (2.10) is satisfied at $u(0) = N$ for some triple $(j,\ k,\ l)$ then the value of l in this triple must be unique. This satisfies part of the condition (iv) of Lemma 2.2. If the set L is not unique then we have the interesting possibility of a mixed mode interaction at the SBB, we shall discuss this in Section 4 with particular reference to the function (1.2).

To proceed we now prove the following result.

Lemma 2.5. *Let the function $\bar{\psi}(\boldsymbol{r})$ be defined as above and let $u(r)$ be a solution of problem* (2.1). *We suppose that* (2.15) *is satisfied by $\bar{\psi}(\boldsymbol{r})$ when $(u,\ \lambda) = (u_0,\ \lambda_0)$. We now define a function $A : C^2[0,1] \times \mathbb{R} \to \mathbb{R}$ as follows:*

$$A(u,\ \lambda) = \int_0^1 \left[f_u(u) + \lambda \frac{\partial u}{\partial \lambda} f_{uu}\right] T_l^2(r) r^2\, \mathrm{d}r. \tag{2.17}$$

Then the condition (ii) *of Theorem* 2.4 *is satisfied iff*

$$A(u_0,\ \lambda_0) \neq 0.$$

Proof. It may be readily shown that the function $M_{w\lambda}\zeta$ takes the following form:

$$M_{w\lambda}\zeta = \left[\Delta^{-1}\left(f_u + \lambda f_{uu}\frac{\partial u}{\partial \lambda}\right)\right]\zeta \equiv \Delta^{-1}(h(r))\zeta. \tag{2.18}$$

Let us now suppose that for some function $\zeta \in K$ there exists a function η such that $M_{w\lambda}\zeta = M_w\eta$. It follows immediately that

$$h(r)\zeta = \Delta\eta + \lambda f_u \eta. \tag{2.19}$$

We may now employ the Fredholm alternative to deduce that we may find such a function η iff the following condition is satisfied.

$$\langle\bar{\psi},\ h(r)\zeta\rangle = 0 \quad \text{for all functions } \bar{\psi} \in K. \tag{2.20}$$

Where $\langle f,\ g\rangle$ is the standard inner product on the space $C^0(B)$.

We may thus verify the condition (ii) of the theorem if for each $\zeta \in K$ there is some function $\bar{\psi} \in K$ such that

$$\int_B \bar{\psi}^* \zeta h(r) r^2 \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi \neq 0, \tag{2.21}$$

where $\bar{\psi}^*$ is the complex conjugate of the function $\bar{\psi}$. Thus, if we take $\bar{\psi} \equiv \zeta$ then

$\bar{\psi}^*\zeta = T_l^2(r)F(\theta, \phi)$ where $F(\theta, \phi)$ is a positive function of θ and ϕ. Hence the condition (2.21) is equivalent to the condition that $A \neq 0$. □

(In Section 3 we shall show that A plays an important role in the study of the bifurcation equations at the SBB.)

To complete our proof of Lemma 2.2 we shall now prove the following result.

Lemma 2.6. *Let the functions $\mu_j(N)$ and $\alpha_{lk}(N)$ be defined as in Lemma 2.2. Let there be an ISB when $N = N_0$, $\lambda = \lambda_0$,*

$$v(s) = u\big(r/\mu_j(N)\big) \quad \text{and} \quad R_l(s) = T_l\big(r/\mu_j(N)\big)$$

If $A(u, \lambda)$ is defined as in (2.17) and, further, if $\mathrm{d}\mu_j(N)/\mathrm{d}N \neq 0$ then

$$A = 0 \quad \textit{iff} \quad \frac{\mathrm{d}\big(\mu_j(N) - \alpha_{lk}(N)\big)}{\mathrm{d}N} = 0.$$

Proof. Let us define functions $P(s)$ and $Q(s)$ as follows.

$$Q(s) \equiv s\partial R_l/\partial s \quad \text{and} \quad P(s) = \partial R_l/\partial N.$$

A simple calculation then shows that these functions satisfy the following ordinary differential equations.

$$LQ \equiv Q_{ss} + \frac{2}{s}Q_s - \frac{l(l+1)Q}{s^2} = -2f_u R_l - sf_{uu}\frac{\partial v}{\partial s}R_l. \tag{2.22}$$

and

$$LP = -sf_{uu}\frac{\partial v}{\partial N}R_l. \tag{2.23}$$

If we further differentiate the identity $R_l(\alpha_{lk}(n)) = 0$ with respect to N it follows that

$$\frac{\mathrm{d}\alpha_{lk}}{\mathrm{d}N}Q(\alpha_{lk}) + \alpha_{lk}P(\alpha_{lk}) = 0. \tag{2.24}$$

We now consider the function $M(s)$ defined as follows.

$$M(s) \equiv \frac{\mathrm{d}\alpha_{lk}}{\mathrm{d}N}Q(s) + \alpha_{lk}P(s).$$

It is clear from (2.24) that

$$M(\alpha_{lk}) = 0$$

and

$$LM = -\left[(2f_u + sf_{uu}\partial v/\partial s)\frac{\mathrm{d}\alpha_{lk}}{\mathrm{d}N} + \alpha_{lk}sf_{uu}\frac{\partial v}{\partial N}\right]R_l.$$

Thus, as $LR_l = 0$ and $R_l(\alpha_{lk}) = 0$ it follows from the Fredholm Alternative that

$$0 = \int_0^{\alpha_{lk}} f_{uu}\left[s\frac{\partial v}{\partial s}\frac{\mathrm{d}\alpha_{lk}}{\mathrm{d}N} + \alpha_{lk}\frac{\partial v}{\partial N}\right]R_l^2 s^2\,\mathrm{d}s + 2\frac{\mathrm{d}\alpha_{lk}}{\mathrm{d}N}\int_0^{\alpha_{lk}} f_u R_l^2 s^2\,\mathrm{d}s. \tag{2.26}$$

We shall now find and expression for A in terms of the function $v(s)$. By the definition of $v(s)$ we have

$$u(r) = v\big(\lambda^{1/2}(N)r\big) = v\big(\mu_j(N)r\big)$$

and thus

$$\frac{\partial u}{\partial \lambda} = \frac{\partial v}{\partial N}\frac{\mathrm{d}N}{\mathrm{d}\lambda} + \tfrac{1}{2}r\lambda^{-1/2}\frac{\partial v}{\partial s}. \tag{2.27}$$

Substituting the above identity into the expression (2.17) defining A and changing variables from r to s we see, after some manipulation that

$$2\mu^3\frac{\mathrm{d}\mu}{\mathrm{d}N}A = \int_0^{\mu} f_{uu}\left[s\frac{\partial v}{\partial s}\frac{\mathrm{d}\mu}{\mathrm{d}N} + \alpha_{lk}\frac{\partial v}{\partial N}\right]R_l^2 s^2\,\mathrm{d}s + 2\frac{\mathrm{d}\mu}{\mathrm{d}N}\int_0^{\mu} f_u R_l^2 s^2\,\mathrm{d}s. \tag{2.28}$$

If we now use the identity (2.26) the expression for A given in (2.28) simplifies to the following.

$$2\mu^3\frac{\mathrm{d}\mu}{\mathrm{d}N}A = \left(\frac{\mathrm{d}\mu}{\mathrm{d}N} - \frac{\mathrm{d}\alpha_{lk}}{\mathrm{d}N}\right)\int_0^{\mu}\left[f_{uu}s\frac{\partial v}{\partial s} + 2f_u\right]R_l^2 s^2\,\mathrm{d}s. \tag{2.29}$$

If we now multiply the ordinary differential equation (2.22) by the function $R_l(s)$ and integrate by parts we may deduce that the integral term in the expression (2.29) equals $\mu^3(\partial R_l/\partial s)^2\ldots$. Thus it follows that

$$2\frac{\mathrm{d}\mu}{\mathrm{d}N}A = \left(\frac{\partial R_l}{\partial s}\right)^2\left(\frac{\mathrm{d}\mu}{\mathrm{d}N} - \frac{\mathrm{d}\alpha_{lk}}{\mathrm{d}N}\right). \tag{2.30}$$

The conclusions of the lemma thus follow. □

(We note that the condition $\mathrm{d}\mu/\mathrm{d}N \neq 0$ is equivalent to the statement that

$$\alpha_{0k}(N) \neq \mu_j(N)$$

which is also equivalent to the condition that the problem (1.1) has no fold bifurcations at this point. It is shown by Budd and Norbury [5] that this is merely a technical condition and in practice does not affect the transversality condition.)

3. Transversality and the pairwise existence of ISB points

The behaviour of a nonsymmetric solution of problem (1.1) at a bifurcation point can be very complex and some interesting examples of this are given by Golubitsky and Schaeffer [9]. In this section we shall briefly survey some of the problems involved in studying such solutions. We shall presume that the problem (1.1) has an ISB when $\lambda = \lambda_0$ such that the function $\bar{\psi}(\boldsymbol{r})$ has the form $T_l(r)Y_{lm}(\theta, \phi)$ for some unique value of l. To examine the nonsymmetric solutions of problem (1.1) in the neighbourhood of the ISB we set

$$u(r, \theta, \phi; \lambda) = u(r; \lambda) + \sum_{m=-l}^{l} a_m(\lambda)Y_{lm}(\theta, \phi)T_l(r) + z(r, \theta, \phi; \lambda) \tag{3.1}$$

where $u(r; \lambda)$ is the solution of the ordinary differential equation problem (2.1) and the function

$z(r, \theta, \phi; \lambda)$ is small if λ is close to λ_0. We now present the reduced bifurcation equations (defined by Sattinger [13]): a set of nonlinear algebraic equations locally satisfied by the coefficients $a_m(\lambda)$ when λ is close to λ_0.

Lemma 3.1. (i) *If l is even the reduced bifurcation equations are*

$$(\lambda - \lambda_0)Aa_m + \tfrac{1}{2}\lambda_0 B \sum_{p+q=m} C_m^{pq} a_p a_q = 0, \tag{3.2}$$

where A is defined by the expression (2.17),

$$B = \int_0^1 f_{uu}(u) T_l^3 r^2 \, \mathrm{d}r$$

and

$$C_m^{pq} = \int_0^{2\pi}\int_0^{\pi} Y_{lp} Y_{lq} Y_{lm}^* \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi. \tag{3.3}$$

(ii) *If l is odd then the coefficients C_m^{pq} vanish and the reduced bifurcation equations contain third order terms.*

(We note that, up to a scaling of the difference $\lambda - \lambda_0$, these equations are independent of the function $f(u)$).

Proof. This is a simple calculation; use the Liapounov–Schmidt reduction technique. □

Corollary 3.2. *The equations* (3.2) *admit an axisymmetric solution with $a_m(\lambda) = 0$ if $m \neq 0$. If l is even the corresponding bifurcation is transcritical and if l is odd then it is a pitchfork.*

(The above result has been derived using more abstract ideas from group theory by Vanderbauwhede [15] and by Ihrig and Golubitsky [11].)

We may now see very clearly that the nature of the bifurcation point changes if $A = 0$ as then the linear term vanishes in the expression (3.2).

To find examples of systems where this behaviour occurs we have numerically investigated the SBB's of problem (1.1) when $f(u) = u(1 + |u|^{p-1})$ and in particular we have studied their behaviour as we vary the value of p. (The details of this calculation will be given in Section 4.)

To do this we proceed as in Section 2 and we compute the two functions $\mu_j(N)$ and $\alpha_{lk}(N)$. As the values of these functions now depend also upon the value of p we shall henceforth denote them by $\mu_j(N; p)$ and $\alpha_{lk}(N; p)$. It may be readily shown that, for fixed N, both $\mu_j(N; p)$ and $\alpha_{lk}(N; p)$ are differentiable functions of p.

Definition 3.1. Let j, k, and l be fixed. We define the set Σ_{jkl} as follows

$$\Sigma_{jkl} = \{(p, N) : \mu_j(N; p) = \alpha_{lk}(N; p)\},$$

hence Σ_{jkl} is the locus of the points (p, N) such that the problem (1.1)–(1.2) has an ISB when $u(0) = N$.

(The calculation of Σ_{jkl} is described in Section 4. In general Σ_{jkl} is a one-dimensional submanifold of the set $\mathbb{R}^2$ but we note that it may not be connected.)

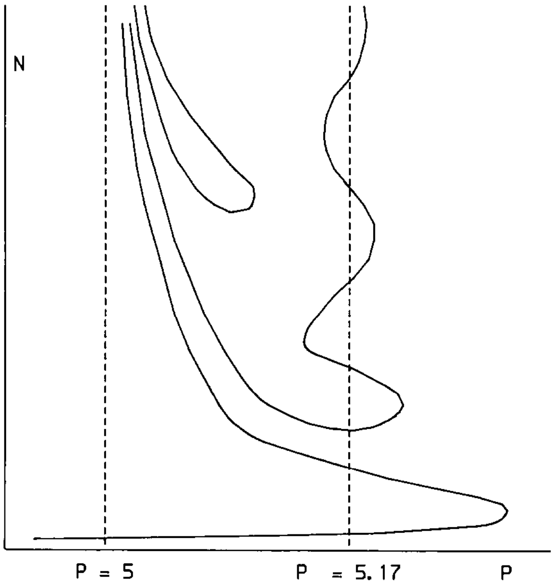

Fig. 1. The set Σ_{221}.

In Fig. 1. we exhibit the set Σ_{221}.

We observe, numerically, that as p varies there are points $(p^*, N^*) \in \Sigma_{jkl}$ such that when $p = p^*$ the curves $\mu_j(N;\ p^*)$ and $\alpha_{lk}(N;\ p^*)$ (considered as functions of N) touch. Further, if $p > p^*$ (or $p < p^*$) the curves do not intersect and if $p < p^*$ (or $p > p^*$) they intersect

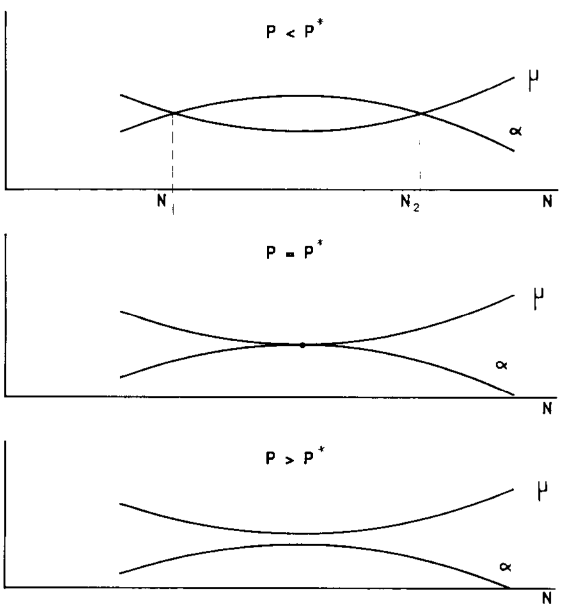

Fig. 2. Transversal and non-transversal intersections of μ and α.

transversally at two points. These points occur when N takes the two values N_1 and N_2 which are close to N^*. When $(p, N) = (p^*, N^*)A = 0$ and if $p < p^*$ (or $p > p^*$) then $A(N_1)A(N_2) < 0$ (where $A(N_i)$ is the value taken by A when $N = N_i$). This behaviour is shown in Fig. 2. We summarise these observations by the following lemma.

Lemma 3.3. *Let the intersection of the set Σ_{jkl} with a neighbourhood of the point (p^*, N^*) be described locally by the functional relation $p \equiv p(N)$. Then if $\mathrm{d}p/\mathrm{d}N = 0$ at the point (p^*, N^*) (and hence if the set Σ_{jkl} has a turning point at (p^*, N^*)) it follows that $A = 0$ at this point.*

Proof. If we differentiate the identity $\mu_j(N;\ p(N)) = \alpha_{lk}(N;\ p(N))$ with respect to N a simple calculation shows that

$$\frac{\mathrm{d}p}{\mathrm{d}N}\left(\frac{\alpha\alpha_{lk}}{\partial p} - \frac{\partial\mu_j}{\partial p}\right) = \frac{\partial\alpha_{lk}}{\partial N} - \frac{\partial\mu_j}{\partial N}.$$

The result then follows from Lemma 2.6. □

It is evident from Fig. 1 that there are many points (p^*, N^*) at which the set Σ_{jkl} has a turning point and hence the transversality condition for an SBB fails. We shall show later in Section 4 that these points accumulate. To be precise, there are two critical values of p namely 5 and p^∞ such that there are two sequences $(p_{i,n}^*, N_{i,n}^*) \in \Sigma_{jkl}$ with $i = 1, 2$ and $n \to \infty$ such that $p_{1,n}^* \to 5$, $p_{2,n}^* \to p^\infty$ and $N_{i,n}^* \to \infty$ as $n \to \infty$. For all points on each sequence $A = 0$ and the transversality condition fails.

If we consider the values of p close to such a p^* it is evident that we have a nonsymmetric solution bifurcating from the symmetric solution only if p is either less than or greater than p^* but not for all values of p in the neighbourhood of p^*. In Fig. 3 we indicate two possible forms

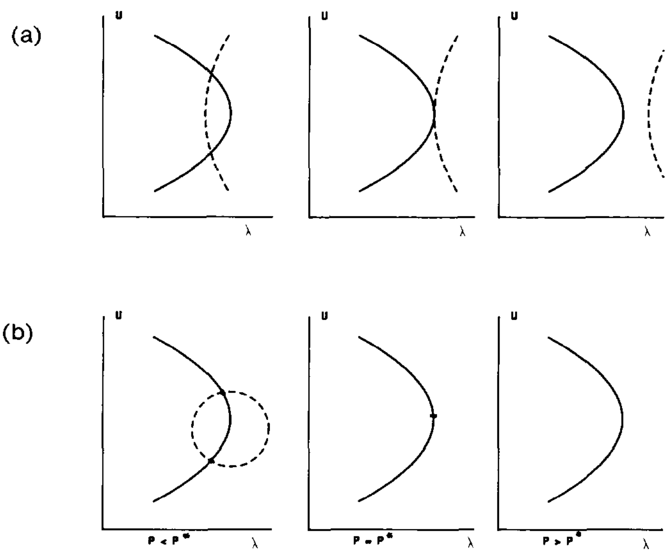

Fig. 3. Examples of bifurcation. The solid line is a symmetric solution and the broken line a non-symmetric solution.

that the resulting non-symmetric solution branch may take. In the case (a) indicated in this figure the point (p^*, N^*) is an SBB, with a non-symmetric solution existing for all p in a neighbourhood of p^* but only bifurcating from the symmetric solution if $p \leqslant p^*$.

In the case (b), (p^*, N^*) is not an SBB and a non-symmetric solution branch only exists if $p < p^*$. It would be interesting to investigate these branches further to determine which outcome does indeed occur.

4. The numerical calculation of ISB points

In this section we shall briefly outline the numerical methods employed to calculate the ISB points for the problem (1.1)–(1.2) and hence to determine the set Σ_{jkl} depicted in Fig. 1. We shall also discuss the ISB points at which we may get mixed mode interactions and those values of $p \equiv p^\infty$ described in Section 3 for which there is an accumulation of ISB points for a fixed value of l.

To calculate the function $\mu_j(N;\ p)$ we numerically integrate the differential equation problem (2.2) using the NAG Runge–Kutta–Merson routine D02BGF. To calculate $\alpha_{lk}(N;\ p)$ it proves numerically convenient to compute the function $K(s)$ where $K(s) = R_l(s)/s^l$ and the function $R_l(s)$ satisfies the differential equation (2.8). A simple calculation then shows that the function $K(s)$ satisfies the following differential equation:

$$K_{ss} + \frac{2}{s}(l+1)K_s + (1 + p\,|v|^{p-1})K = 0, \tag{4.1}$$

$$K(0) = 1 \quad \text{and} \quad K_s(0) = 0,$$

where the function $v(s)$ is the solution of problem (2.2) with $v(0) = N$. Problem (4.1) may then be numerically integrated as above.

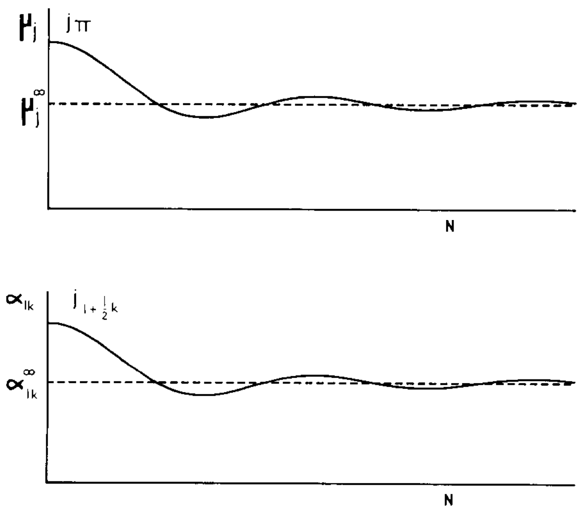

Fig. 4. The curves $\mu(N)$ and $\alpha(N)$.

In Fig. 4 we indicate the resulting graphs of $\mu_j(N;\ p)$ and $\alpha_{lk}(N;\ p)$ expressed as functions of N with p fixed. We note that these graphs are qualitatively similar. Indeed there are values μ_j^∞ and α_{lk}^∞ such that

$$\mu_j(N;\ p) \to \mu_j^\infty \quad \text{and} \quad \alpha_{lk}(N;\ p) \to \alpha_{lk}^\infty \quad \text{as } N \to \infty,$$

moreover both $\mu_j(N;\ p)$ and $\alpha_{lk}(N;\ p)$ oscillate about these asymptotic values. (The oscillation is closely related to a dynamical system associated with the problem (1.1)–(1.2) and it is discussed in detail by Budd [3].) We shall now make the following important observation which is supported both by the above numerical evidence and some asymptotic calculations reported by Budd [1].

Conjecture 4.1. *The following inequality holds for all values of N, p, j, k and l.*

$$|\alpha_{lk}(N;\ p) - \alpha_{lk}^\infty| \ll |\mu_j(N;\ p) - \mu_j^\infty|. \tag{4.2}$$

(Indeed the numerical evidence suggests that the above two terms are in the ratio of about 1 to 50.)

Further, if $N = 0$ and thus $v(s) \equiv 0$, the equation (4.1) simplifies to a form of Bessels equation with the solution

$$K(s) = s^{-(l+1/2)} J_{l+1/2}(s)$$

where $J_\nu(s)$ is the νth Bessel function which is regular at the origin. Thus

$$\alpha_{lk}(0;\ p) = j_{l+1/2,k} \tag{4.3}$$

where $j_{l+1/2,k}$ is the kth positive zero of $J_{l+1/2}(s)$. Combining (4.2) and (4.3) we conjecture that the following inequality holds for all values of N, p, j, k and l.

$$|\alpha_{lk}(N;\ p) - j_{l+1/2,k}| \ll |\mu_j(N;\ p) - \mu_j^\infty|. \tag{4.4}$$

Hence the value of $\alpha_{lk}(N;\ p)$ is approximately a constant independent of the values of N and p. Thus having determined the values of $\mu_j(N;\ p)$ we may readily estimate the location of the ISB points by solving the equation

$$\mu_j(N) = j_{l+1/2,k}.$$

With this approximation in mind it is clear that the points at which the curve $\alpha_{lk}(N;\ p)$ (regarded as a function of N) intersects the curve $\mu_j(N;\ p)$ non-transversally must occur close to the limit points of the latter curve.

To numerically locate the ISB points for problem (1.1)–(1.2) we must solve the equation

$$F(N;\ p) \equiv K(\mu_j(N;\ p)) = 0 \tag{4.5}$$

and hence determine the set Σ_{jkl}. It is evident from the form of Σ_{jkl} described in Fig. 1 that this set has a number of limit points and hence a pseudo arc-length method was employed to solve problem (4.5). To use such a technique we require the values of the partial derivatives of the scalar function $F(N;\ p)$ with respect to the parameters N and p. To determine $\partial F/\partial N$ we note that

$$\frac{\partial F}{\partial N} = \frac{\partial K}{\partial N}(\mu_j) + \frac{\partial \mu_j}{\partial N} K_s(\mu_j)$$

$$0 = \frac{\partial v}{\partial N}(\mu_j) + \frac{\partial \mu_j}{\partial N} v_s(\mu_j).$$

Hence

$$\frac{\partial F}{\partial N} = \left[\frac{\partial K}{\partial N} - \frac{\partial v}{\partial N} \frac{\partial K}{\partial s} \Big/ \left(\frac{\partial v}{\partial s} \right)^{-1} \right] \Bigg|_{\mu_j}. \tag{4.6}$$

The functions $\partial K/\partial N$ and $\partial v/\partial N$ may be readily calculated by differentiating the expressions (2.2) and (4.1) with respect to N and then numerically integrating the resulting linear ordinary differential equations. A very similar calculation allows us to obtain a value for $\partial F/\partial p$. Hence we may implement the pseudo arc-length algorithm to determine Σ_{jkl} in the form described by Keller [12].

A graph of Σ_{jkl} such as that given in Fig. 1 may thus be determined for each value of the triple (j, k, l) given in Example 2.2. We shall now discuss three important features of this figure.

4.1. The behaviour of Σ_{jkl} as $p \to 5$

The figures obtained for each triple (j, k, l) (with $l \neq 0$) have qualitatively very similar behaviour as $p \to 5$ from above. We find that as p approaches 5 (the critical Sobolev exponent for $\mathbb{R}^3$) the curve Σ_{jkl} comprises many separate components. These consist of curves $(p, N_i(p))$ such that

$$N_i(p) < N_{i+1}(p) \quad \text{and} \quad N_i(p) \to \infty \quad \text{as } i \to \infty.$$

Further careful calculations show that there are constants $C_i(j, k, l)$ such that as $p \to 5$

$$N_i^4(p) \sim C_i(p-5)^{-2(i-1)}.$$

Thus $N_1(p)$ is a continuous function of p for all values of p in an open interval containing 5 wheras, if $i > 1$ then $N_i(p) \to \infty$ as $p \to 5$ from above. (The existence of an ISB point for the special case of $i = 1$ and $p = 5$ has been rigorously established by Budd and Norbury [5]. As yet there is no fully rigorous proof of the existence of the other ISB points described above.)

It is of interest to note that if $l = 0$ (and hence if we are considering a fold bifurcation point) it may be shown by the formal asymptotic methods described by Budd [2,4] that the ith fold bifurcation point (ordered as before) occurs when

$$N_i(p)^4 \sim D_i(p-5)^{1-2i} \quad \text{as } p \to 5$$

where D_i is a constant which can be explicitly calculated. Thus, as $p \to 5$ the ISB points lie between the fold bifurcations.

4.2. Mixed mode interactions

These occur if there are values N, p such that for some values of j, k, l and k', l' with $l \neq l'$ and $k \neq k'$ the following identity holds.

$$\mu_j(N) = \alpha_{lk}(N;\ p) = \alpha_{l'k'}(N;\ p). \tag{4.7}$$

Hence $(p, N) \in \Sigma_{jkl} \cap \Sigma_{jk'l'}$. No such points have been observed numerically for the values of j, k and l given in Example 2.2. Indeed our previous calculations imply that the existence of such points is unlikely. This conclusion follows from the observation (4.4) that the values of $\alpha_{lk}(N;\ p)$ and $\alpha_{l'k'}(N;\ p)$ are almost independent of N and p. and hence we are unlikely to

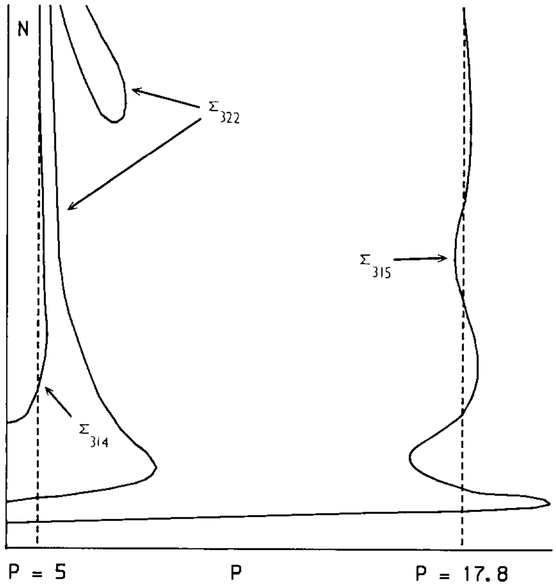

Fig. 5. A comparison of the curves Σ_{314}, Σ_{322} and Σ_{315}.

observe a mixed mode interaction unless the values of $j_{l+1/2,k}$ and $j_{l'+1/2,k'}$ are very close. We present in Fig. 5 a comparison of part of the three curves Σ_{314}, Σ_{322} and Σ_{315}. The corresponding values of $j_{l+1/2,k}$ are as follows.

$$j_{9/2,1} = 8.183\ldots, \qquad j_{5/2,2} = 9.095\ldots, \qquad j_{11/2,1} = 9.355\ldots \ . \tag{4.8}$$

(For comparison $5\pi/2 = 7.854...$ and $3\pi = 9.425...$.) It is evident from Fig. 5 that the three above curves do not intersect. Indeed we may deduce some of their their features by considering the intersection of the graph of the function $\mu_3(N)$ presented in Fig. 4 with the three lines $\alpha_{l,k}(N) \approx j_{l+1/2,k}$ with the values of $j_{l+1/2,k}$ given as in (4.8).

Conjecture 4.2. *Let the triples* (j, k, l) *and* (j', k', l') *be ordered so that*

$$j_{l+1/2,k} < j_{l'+1/2,k'}. \tag{4.9}$$

Now, let p *be fixed in value and let* $N_{kl}(p)$ *be the smallest positive value of* N *such that* $(p, N_{kl}(p)) \in \Sigma_{3kl}$. *Then*

$$N_{k'l'} < N_{kl}. \tag{4.10}$$

"Proof" of Conjecture 4.2: We shall employ the approximation that $\alpha_{lk}(N) = j_{l+1/2,k}$. Further asymptotic and numerical evidence implies that the curve $\mu_3(N)$ oscillates about a point μ_3^∞ with the magnitude of the oscillations decreasing as the value of N increases. Thus, the first intersection of $\mu_3(N)$ with the curve $\alpha_{lk}(N) = j_{j+1/2,k}$ must occur on that portion of the curve $\mu_3(N)$ for which $\mathrm{d}\mu_3(N)/\mathrm{d}N < 0$. The result (4.10) therefore follows from the above observation and the condition (4.9).

Similar results may be conjectured for subsequent values of N for each fixed value of p.

4.3. Accumulation of ISB points when $p = p^{\infty}$

As we observed in Section 3 there are values of $p = p^{\infty}$ such that the point (p^{∞}, ∞) is an accumulation point for the curve Σ_{jkl}. It is shown by Budd [1] that this occurs if the values of $\mu_j^{\infty}(p)$ and $\alpha_{lk}^{\infty}(p)$ coincide. Where the values of μ_j^{∞} and α_{lk}^{∞} are defined as in Corollary 2.3 but we now make explicit reference to their dependence upon p. It is further shown that $\mu_j^{\infty}(p)$ is the jth positive zero of the function $M(s)$ where $M(s)$ satisfies the differential equation (2.2) together with the following singular initial condition.

$$s^{\gamma}M(s) \to \left[\gamma(1-\gamma)\right]^{1/(p-1)}$$

where $\gamma = 2/(p-1)$. Moreover, if we substitute the function $M(s)$ for the function $v(s)$ in the differential equation (2.8) then the zeros of the resulting solution of the corresponding initial value problem are precisely the values $\alpha_{lk}^{\infty}(p)$ given above. Evidently the values of $\mu_j^{\infty}(p)$ and $\alpha_{lk}^{\infty}(p)$ may be calculated as p is varied and it is then straightforward to find values of $p = p^{\infty}$ at which they coincide. We note, however, that such values of p^{∞} depend upon the triple (j, k, l) and for some values of this triple there is no corresponding p^{∞}.

We tabulate below some values of p^{∞}.

j	k	l	p^{∞}
2	1	2	5.17297
3	1	5	17.85415
3	2	2	6.72180
4	3	3	8.05462

An interesting problem for future research would be a study of problem (1.1)–(1.2) with exponent values close to those given above.

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